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A class of algebraically general non-null Einstein–Maxwell fields II

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Abstract. A class of solutions of the Einstein–Maxwell field equations which satisfy the condition that the null tetrad determined by the Maxwell bivector is parallelly transported along m^i and $\bar{m}^i$ is investigated. It is shown that the vector m_i must be twist-free. It is also shown for fields satisfying the dual condition that the null tetrad is parallelly propagated along l^i and n^i, that the vectors l^i and n^i either both have vanishing twist or vanishing divergence. Consequently the exact solutions found earlier by Tupper and his collaborators and by the author are in fact the most general solutions. A complex coordinate transformation relating the two types of twist-free solution is given.

1. Introduction

In an earlier paper (Barnes 1976, to be referred to as I) I investigated non-null Einstein–Maxwell fields which satisfied the condition that the principal null tetrad associated with the electromagnetic field was parallelly transported along the complex-conjugate null vectors m_i and $\bar{m}_i$ or equivalently along any direction lying in the two-dimensional space-like eigenblade of the Maxwell bivector. It was shown that no solutions exist for which m_i is divergence-free and the general solution for the case where m_i is twist-free was found. In this paper I give an alternative characterization of the fields and show that no solutions exist other than those already found in I.

Attention is then turned to electromagnetic fields satisfying the dual condition that the principal null tetrad is parallelly transported along directions lying in the time-like eigenblade of the Maxwell bivector. The field equations have been integrated in the twist-free case by Tariq and Tupper (1975) and in the divergence-free case by McLenaghan and Tariq (1975) and independently by Tupper (1976). In this paper I show that these two solutions are in fact the only fields belonging to this class. This result has been partially proved by McLenaghan and Tariq (1976). The two investigations share the common feature that a re-scaled tetrad is used which is only weakly parallelly propagated along l^i and n^i. However, whereas the method of McLenaghan and Tariq is applicable only to the case $\rho = \mu$ and a different approach is necessary for the case $\rho = -\mu$, the method used in this paper leads to essentially the same proof for both cases. Finally, I show how the twist-free solutions for the two classes of field are related by a complex coordinate transformation.

2. The space-like case

As in I, I employ the spin coefficient formalism and notation of Newman and Penrose (1962, to be referred to as NP). The self-dual Maxwell bivector F^+_{ij} can be written in the form

$$F^+_{ij} = \phi(n_{[i}l_{j]} + m_{[i}\bar{m}_{j]}), \tag{2.1}$$

where ϕ is the complex field strength and l^i, n^i, m^i and $\bar{m}^i$ are the four principal null vectors associated with the electromagnetic field and satisfy the usual orthonormality relations. Equation (2.1) determines the null tetrad only up to transformations of the form

$$\tilde{l}^i = Al^i, \qquad \tilde{n}^i = A^{-1}n^i, \qquad \tilde{m}^i = m^i\,e^{i\theta}. \tag{2.2}$$

It is assumed that the tetrad can be chosen so that it is parallelly transported along m_i and $\bar{m}_i$, i.e.

$$l_{i;j}m^j = n_{i;j}m^j = m_{i;j}m^j = \bar{m}_{i;j}m^j = 0 \tag{2.3}$$

or equivalently

$$\rho = \sigma = \mu = \lambda = \alpha = \beta = 0, \tag{2.4}$$

where ρ, σ, etc are spin coefficients defined in NP.

Theorem. The existence of a principal null tetrad satisfying equation (2.3) is equivalent to the validity of the conditions

$$F^+_{ij;k}m^k = fF^+_{ij}, \qquad F^+_{ij;k}\bar{m}^k = gF^+_{ij} \tag{2.5}$$

and

$$2C_{ijkl}l^in^j(l^kn^l - m^k\bar{m}^l) = R_{ij}(l^in^j + m^i\bar{m}^j) \tag{2.6}$$

for any principal null tetrad in the equivalence class defined by (2.2).

Proof. Equations (2.1) and (2.3) imply (2.5) with $f = (\ln\phi)_{,k}m^k$ and $g = (\ln\phi)_{,k}\bar{m}^k$. If the conditions (2.4) are substituted into equation (4.2l) of NP one obtains $\phi_{11} - \psi_2 = 0$ which is the spin coefficient version of (2.6). Conversely by contracting equation (2.5) with l^im^j and $n^i\bar{m}^j$ one can deduce that

$$\sigma = \rho = \lambda = \mu = 0. \tag{2.7}$$

Equation (2.7) is invariant under the tetrad transformations (2.2) whereas α and β transform as follows:

$$\tilde{\alpha} = e^{-i\theta}(\alpha + \tfrac{1}{2}\bar{\delta}W); \qquad \tilde{\beta} = e^{i\theta}(\beta + \tfrac{1}{2}\,\delta W),$$

where $W = \ln A + i\theta$. Hence if $\bar{\delta}W = -2\alpha$ and $\delta W = -2\beta$ are consistent, the result is proved. If we apply the commutator $\delta\bar{\delta} - \delta\bar{\delta}$ to W and use (4.2l) and (4.4) of NP we see that the integrability condition is $\psi_2 - \phi_{11} = 0$ which is simply (2.6).

A tetrad satisfying equation (2.7) will be called weakly parallelly propagated along m^i and $\bar{m}^i$.

In the following analysis it will be convenient to use a re-scaled tetrad which is only weakly parallelly propagated and to use the freedom inherent in (2.2) to impose more useful restrictions on the spin coefficients at a later stage.

Maxwell's equations can be written in the form

$$D\phi = \Delta\phi = (\delta - 2\tau)\phi = (\bar{\delta} + 2\pi)\phi = 0. \quad (2.8)$$

From (2.7) and (4.2 k, m) of NP we can deduce

$$\psi_1 = \psi_3 = \psi_2 - \phi\bar{\phi} = 0. \quad (2.9)$$

The Bianchi identities take the form

$$\kappa\psi_4 = (2\bar{\tau} - \pi)\psi_2 \quad (2.10a)$$

$$\nu\psi_0 = (2\bar{\pi} - \tau)\psi_2 \quad (2.10b)$$

$$(D + 4\epsilon)\psi_4 = 0 \quad (2.10c)$$

$$(\Delta - 4\gamma)\psi_0 = 0 \quad (2.10d)$$

$$(\delta - \tau + 4\beta)\psi_4 = -\psi_2\nu \quad (2.10e)$$

$$(\bar{\delta} + \pi - 4\alpha)\psi_0 = \psi_2\kappa. \quad (2.10f)$$

Applying the commutators to ϕ we can deduce with the aid of Maxwell's equations and (4.2c, i) of NP that

$$\tau\bar{\tau} = \pi\bar{\pi} \quad (2.11a)$$

$$D\tau = (\epsilon - \bar{\epsilon})\tau \quad (2.11b)$$

$$D\pi = -(\epsilon - \bar{\epsilon})\pi \quad (2.11c)$$

$$\Delta\tau = (\gamma - \bar{\gamma})\tau \quad (2.11d)$$

$$\Delta\pi = -(\gamma - \bar{\gamma})\pi \quad (2.11e)$$

$$D\nu = -(3\epsilon + \bar{\epsilon})\nu \quad (2.11f)$$

$$\Delta\kappa = (3\gamma + \bar{\gamma})\kappa. \quad (2.11g)$$

If we operate on equations (2.10a, b) and (2.11a) with D, Δ and δ respectively we obtain

$$D\kappa = (3\epsilon + \bar{\epsilon})\kappa \quad (2.12a)$$

$$\Delta\nu = -(3\gamma + \bar{\gamma})\nu \quad (2.12b)$$

$$(\tau + \bar{\pi})(\psi_2 + 2\tau\bar{\tau}) - \tau\bar{\nu}\bar{\kappa} - \bar{\pi}\nu\kappa - (\bar{\tau} + \pi)\kappa\bar{\nu} = 0. \quad (2.12c)$$

The cases $\tau + \bar{\pi} = 0$ and $\tau - \bar{\pi} = 0$ were considered in I and so below we will assume that neither of these conditions holds (i.e. $\tau\pi \neq \bar{\tau}\bar{\pi}$). By a suitable choice of θ in (2.2) it is possible to set $\tau = \pi$ and hence (2.12c) reduces to $\kappa\bar{\nu} = \bar{\kappa}\nu$. Consequently κ^2 and ν^2 have the same argument and A in (2.2) may be chosen so that $\kappa = \pm\nu$. The tetrad is now completely determined apart from ambiguities in sign.

From equations (2.10)–(2.12) we can deduce that $\epsilon = \gamma = 0$ and either $\kappa = \nu$, $\psi_0 = \psi_4$ or $\kappa = -\nu$, $\psi_0 = -\psi_4$. Hence from (2.10), (2.11) and (4.2c, d, o, r) of NP we have

$$D\alpha = \Delta\alpha = D\beta = \Delta\beta = D\psi_0 = \Delta\psi_0 = D\psi_4 = \Delta\psi_4 = 0.$$

The commutator $\Delta D - D\Delta$ applied to any of the spin coefficients α, β, τ, κ, ψ_0 and ψ_4 vanishes and consequently from (4.4) of NP the same is true of the operator $\delta + \bar{\delta}$. In particular applying this commutator to ψ_0 we deduce with the aid of (2.10e, f) that

$\alpha = \beta$. Finally (4.2l) of NP reduces to the form

$$\delta\alpha = \alpha(\bar{\alpha} - \alpha). \tag{2.13}$$

To summarize: we have seen that the null tetrad may be chosen so that

$$\rho = \sigma = \lambda = \mu = \epsilon = \gamma = \alpha - \beta = 0 \tag{2.14a}$$

and

$$\tau = \pi, \qquad \kappa = \pm\nu, \qquad \psi_0 = \pm\psi_4. \tag{2.14b}$$

Furthermore we have

$$\mathrm{D}x = \Delta x = (\delta + \bar{\delta})x = 0 \tag{2.14c}$$

where x is one of the spin coefficients τ, κ, α, ψ_0. Two cases arise depending on whether α is purely real or not.

Case (*i*). $\alpha - \bar{\alpha} = 0$

Equations (2.13) and (2.14) imply that α is a real constant. The vectors m_i and $\bar{m}_i$ have vanishing Lie bracket and consequently coordinates may be chosen so that (Eisenhart 1925)

$$m^i \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}.$$

By applying the commutators $\delta\mathrm{D} - \mathrm{D}\delta$ and $\delta\Delta - \Delta\delta$ to the coordinate functions x^i and taking real and imaginary parts we obtain

$$\frac{\partial}{\partial x}\begin{pmatrix} l^i \\ n^i \end{pmatrix} = \mathbf{A}\begin{pmatrix} l^i \\ n^i \end{pmatrix} \tag{2.15}$$

$$\frac{\partial}{\partial y}\begin{pmatrix} l^i \\ n^i \end{pmatrix} = \mathbf{B}\begin{pmatrix} l^i \\ n^i \end{pmatrix} \tag{2.16}$$

where $\mathbf{A}$ and $\mathbf{B}$ are real 2×2 matrices such that

$$\mathbf{A} + \mathrm{i}\mathbf{B} = \begin{pmatrix} 2\alpha - \bar{\tau} & \kappa \\ -\bar{\nu} & \tau - 2\alpha \end{pmatrix}.$$

$\mathbf{A}$ and $\mathbf{B}$ are independent of x as a consequence of (2.14c). On integrating (2.15) we obtain

$$\begin{pmatrix} l^i \\ n^i \end{pmatrix} = \exp(x\mathbf{A})\begin{pmatrix} l^i_0 \\ n^i_0 \end{pmatrix} \tag{2.17}$$

where l^i_0 and n^i_0 are independent of x. Exp($x\mathbf{A}$) is non-singular as l^i and n^i are linearly independent. On differentiating (2.17) with respect to y and using (2.16) we can deduce that

$$\left(x\frac{\partial \mathbf{A}}{\partial y} - \mathbf{B} + \mathbf{I}\frac{\partial}{\partial y}\right)\begin{pmatrix} l^i_0 \\ n^i_0 \end{pmatrix} = 0. \tag{2.18}$$

Only the first term in the preceding equation is dependent on x and consequently $\partial\mathbf{A}/\partial y = 0$. This implies that $\partial(\tau + \bar{\tau})/\partial y = 0$, but from equation (4.2$p$) of NP we see that $\partial(\tau + \bar{\tau})/\partial y = -\mathrm{i}(\tau^2 - \bar{\tau}^2)$. Hence $\tau^2 - \bar{\tau}^2 = 0$ which is a contradiction as $\tau = \pi$ and we assumed initially that $\tau\pi \neq \bar{\tau}\bar{\pi}$.

Case (ii). $\alpha-\bar{\alpha}\neq 0$

In this case

$$m^i{}_{,j}\bar{m}^j-\bar{m}^i{}_{,j}m^j=(\alpha-\bar{\alpha})(m^i+\bar{m}^i).$$

Consequently $m^i+\bar{m}^i$ and $m^i-\bar{m}^i$ are two-surface forming (Flanders 1963) and coordinates may be chosen so that

$$m^i\frac{\partial}{\partial x^i}=P\frac{\partial}{\partial x}+(M+\mathrm{i})\frac{\partial}{\partial y}, \tag{2.19}$$

where P and M are real functions such that

$$\frac{\partial P}{\partial y}=\mathrm{i}(\alpha-\bar{\alpha})P, \qquad \frac{\partial M}{\partial y}=\mathrm{i}(\alpha-\bar{\alpha})M. \tag{2.20}$$

As $\delta-\bar{\delta}=2\mathrm{i}\,\partial/\partial y$, we may deduce from (2.13) that $\partial(\alpha-\bar{\alpha})/\partial y=\mathrm{i}(\alpha-\bar{\alpha})^2$, which on integration, gives $\alpha-\bar{\alpha}=\mathrm{i}/y$. The arbitrary function of integration has been absorbed into y by means of a coordinate transformation of the form $\tilde{y}=y+f_0(x^i)$. Here and in what follows a quantity with a zero subscript is independent of y and consequently the transformation preserves the form of (2.19). The integration of (2.13) and (2.20) can now be completed to give

$$\begin{aligned}\alpha&=\frac{\alpha_0+\mathrm{i}}{2y}\\ P&=P_0/y, \qquad M=M_0/y.\end{aligned} \tag{2.21}$$

P_0 is non-zero as the real and imaginary parts of m^i are linearly independent and we can therefore put $P_0=1$ by means of an allowable coordinate transformation of the form $\tilde{x}=g_0(x^i)$. As $(\delta+\bar{\delta})\alpha=0$ we have $y\alpha_{0,x}-M_0(\alpha_0+\mathrm{i})=0$, and consequently $M_0=\alpha_{0,x}=0$. In a similar manner from $\mathrm{D}\alpha=\Delta\alpha=0$ we can deduce that $l^y=n^y=0$ and α_0 is a constant.

Proceeding as in case (i) we can deduce equations identical in form to (2.15)–(2.18) but with

$$\mathbf{A}=\tfrac{1}{2}\begin{pmatrix}2\alpha_0-(\tau+\bar{\tau})y & (\kappa+\bar{\kappa})y\\ -(\nu+\bar{\nu})y & (\tau+\bar{\tau})y-2\alpha_0\end{pmatrix}$$

$$\mathbf{B}=-\tfrac{1}{2}\begin{pmatrix}\tau-\bar{\tau} & \kappa-\bar{\kappa}\\ \nu-\bar{\nu} & \tau-\bar{\tau}\end{pmatrix}$$

As before we obtain $\partial\mathbf{A}/\partial y=0$ which implies

$$\frac{\partial}{\partial y}[(\tau+\bar{\tau})y]=\frac{\partial}{\partial y}[(\kappa+\bar{\kappa})y]=0.$$

From equations $(4.2a, p)$ of NP we can deduce that $\tau-\bar{\tau}=-2\mathrm{i}y^{-1}$ and that either $\kappa=\bar{\kappa}$ or $\tau+\bar{\tau}=2\alpha_0y^{-1}$. In the former case $\kappa=\kappa_0y^{-1}$ where κ_0 is a real constant and equation $(4.2a, p)$ of NP are only consistent if $\alpha_0=0$, $\kappa=\nu$ and $\tau=[\sqrt{(\kappa_0^2-1)}-\mathrm{i}]y^{-1}$. In the latter case $\tau=y^{-1}(\alpha_0-\mathrm{i})=2\bar{\alpha}$. Equations $(4.2a, p)$ of NP and (2.13) of this paper are consistent only if $\kappa=\nu=\pm y^{-1}(1+\mathrm{i}\alpha_0)$.

From equations $(4.2h, p)$ of NP we may deduce that

$$\psi_2=\kappa^2+\kappa\bar{\kappa}-\tau^2-\tau\bar{\tau}+2\tau(\bar{\alpha}-\alpha).$$

On substituting the expressions for τ, κ and α obtained in the preceding paragraph we obtain in the former case $\psi_2=0$ and in the latter $\psi_2=-2y^{-2}(\alpha_0^2-\mathrm{i}\alpha_0)$. From (2.9) we know that ψ_2 is real and positive and so in each case a contradiction is obtained.

Thus there are no fields with $\tau\pi\neq\bar{\tau}\bar{\pi}$. In I the field equations for the case $\tau+\bar{\pi}=0$ (m_i twist-free) were integrated completely and the case $\tau=\bar{\pi}$ (m_i divergence-free) was shown to be inconsistent. Consequently the exact solutions given in I are the only fields satisfying (2.3).

3. The time-like case

Attention is now turned to fields satisfying the dual condition that the principal null tetrad associated with the electromagnetic field is parallelly propagated along l^i and n^i. This condition is equivalent to $\pi=\tau=\kappa=\nu=\epsilon=\gamma=0$, but as in § 2, we will employ only a weakly parallelly propagated tetrad for which

$$\pi=\tau=\kappa=\nu=0. \tag{3.1}$$

We proceed in a manner completely analogous to § 2 replacing formally l^i by m^i and n^i by $-\bar{m}^i$ and vice versa as described in I. We assume that $\rho^2\neq\bar{\rho}^2$ and $\mu^2\neq\bar{\mu}^2$.

Corresponding to equations (2.8)–(2.12) we have

$$(\mathrm{D}-2\rho)\phi=(\Delta+2\mu)\phi=\delta\phi=\bar{\delta}\phi=0 \tag{3.2}$$

$$\psi_1=\psi_3=\psi_2+\phi\bar{\phi}=0 \tag{3.3}$$

$$\lambda\psi_0=-(\rho+2\bar{\rho})\psi_2 \tag{3.4a}$$

$$\sigma\psi_4=-(\mu+2\bar{\mu})\psi_2 \tag{3.4b}$$

$$(\delta+4\beta)\psi_4=0 \tag{3.4c}$$

$$(\bar{\delta}-4\alpha)\psi_0=0 \tag{3.4d}$$

$$(\mathrm{D}-\rho+4\epsilon)\psi_4=-\lambda\psi_2 \tag{3.4e}$$

$$(\Delta+\mu-4\gamma)\psi_0=\sigma\psi_2 \tag{3.4f}$$

$$\rho\bar{\mu}=\bar{\rho}\mu \tag{3.5a}$$

$$\delta\rho=\rho(\bar{\alpha}+\beta) \tag{3.5b}$$

$$\bar{\delta}\rho=\rho(\alpha+\bar{\beta}) \tag{3.5c}$$

$$\delta\mu=-\mu(\bar{\alpha}+\beta) \tag{3.5d}$$

$$\bar{\delta}\mu=-\mu(\alpha+\bar{\beta}) \tag{3.5e}$$

$$\bar{\delta}\sigma=\sigma(3\alpha-\bar{\beta}) \tag{3.5f}$$

$$\delta\lambda=\lambda(\bar{\alpha}-3\beta) \tag{3.5g}$$

$$\delta\sigma=\sigma(3\beta-\bar{\alpha}) \tag{3.6a}$$

$$\bar{\delta}\lambda=\lambda(\bar{\beta}-3\alpha) \tag{3.6b}$$

$$(\rho-\bar{\rho})(\psi_2+2\rho\bar{\mu})+(\bar{\mu}-\mu)\sigma\bar{\sigma}+\rho\bar{\sigma}\bar{\lambda}-\bar{\rho}\sigma\lambda=0 \tag{3.6c}$$

$$(\mu-\bar{\mu})(\psi_2+2\bar{\rho}\mu)+(\bar{\rho}-\rho)\lambda\bar{\lambda}+\mu\bar{\sigma}\bar{\lambda}-\bar{\mu}\sigma\lambda=0. \tag{3.6d}$$

From (3.5*a*) we see that by a suitable choice of A in (2.2) we may put $\rho=\omega\mu$ where $\omega=\pm1$. Hence we may deduce from (3.6*c*, *d*) that $\sigma\bar{\sigma}=\lambda\bar{\lambda}$ and by a suitable choice of θ we can arrange that $\sigma=\omega\lambda$. We can now deduce that

$$\rho=\omega\mu,\qquad \sigma=\omega\lambda,\qquad \epsilon=\omega\gamma,\qquad \psi_0=\psi_4,\qquad \alpha=\beta=0, \tag{3.7a}$$

and

$$(\mathrm{D}+\omega\Delta)x=\delta x=\bar{\delta}x=0,\qquad \mathrm{D}\epsilon=-(\epsilon+\bar{\epsilon})\epsilon \tag{3.7b}$$

where x is one of the spin coefficients ρ, σ, ϵ, ψ_0.

Case (*i*). $\epsilon+\bar{\epsilon}=0$

In this case $\epsilon=\mathrm{i}\epsilon_0$, where ϵ_0 is a real constant. As l^i and n^i have vanishing Lie bracket, coordinates u and v may be chosen so that

$$l^i\frac{\partial}{\partial x^i}=\frac{1}{2}\left(\frac{\partial}{\partial v}+\frac{\partial}{\partial u}\right),\qquad n^i\frac{\partial}{\partial x^i}=\frac{1}{2}\omega\left(\frac{\partial}{\partial v}-\frac{\partial}{\partial u}\right).$$

Applying the commutators $\delta\mathrm{D}-\mathrm{D}\delta$ and $\delta\Delta-\Delta\delta$ to the coordinate functions x^i we obtain

$$\frac{\partial}{\partial v}\begin{pmatrix}U^i\\V^i\end{pmatrix}=\mathbf{A}\begin{pmatrix}U^i\\V^i\end{pmatrix} \tag{3.8}$$

and

$$\frac{\partial}{\partial u}\begin{pmatrix}U^i\\V^i\end{pmatrix}=\mathbf{B}\begin{pmatrix}U^i\\V^i\end{pmatrix} \tag{3.9}$$

where U^i and V^i are the real and imaginary parts of m^i and $\mathbf{A}$ and $\mathbf{B}$ are real 2×2 matrices which are independent of v. By the argument of §2 we can deduce that $\partial\mathbf{A}/\partial u=0$, but

$$\mathbf{A}=2\begin{pmatrix}0 & \mathrm{Im}(\rho+\sigma)-2\epsilon_0\\ \mathrm{Im}(\sigma-\rho)+2\epsilon_0 & 0\end{pmatrix}.$$

Consequently $\partial(\rho-\bar{\rho})/\partial u=0$ and from (3.7*b*) we deduce $\mathrm{D}(\rho-\bar{\rho})=0$ and (4.2*a*) of NP then implies $\rho^2-\bar{\rho}^2=0$ which is a contradiction.

Case (*ii*). $\epsilon+\bar{\epsilon}\neq0$

By methods similar to § 2 case (ii), we can show that coordinates exist such that

$$l^i\frac{\partial}{\partial x^i}=\frac{\partial}{\partial u},\qquad n^i\frac{\partial}{\partial x^i}=\omega\left(u^{-1}\frac{\partial}{\partial v}-\frac{\partial}{\partial u}\right),\qquad \epsilon=\frac{\mathrm{i}\epsilon_0+1}{2u},$$

where ϵ_0 is a real constant. In this coordinate system the u-component of m^i vanishes. Again we obtain equations identical in form with (3.8) and (3.9) but in this case

$$\mathbf{A}=2\begin{pmatrix}0 & u\,\mathrm{Im}(\sigma+\rho)-\epsilon_0\\ u\,\mathrm{Im}(\sigma-\rho)+\epsilon_0 & 0\end{pmatrix}.$$

As $\partial \mathbf{A}/\partial u = 0$, we have

$$\frac{\partial}{\partial u}[(\rho - \rho)u] = \frac{\partial}{\partial u}[(\sigma - \tilde{\sigma})u] = 0.$$

Proceeding in a manner completely analogous to § 2 from equations (4.2*a*, *g*, *h*) of NP and (3.7*b*) we can obtain a contradiction. Thus we have seen that the only solutions that exist have l^i twist-free or l^i divergence-free. In the former case the field equations have been integrated by Tariq and Tupper (1975) and in the latter by McLenaghan and Tariq (1975) and Tupper (1976). In fact by a slight extension of the argument above one can show that the twist-free solutions belong to the class $\rho = -\mu$ whereas the divergence-free solutions all have $\rho = \mu$.

4. A complex coordinate transformation

The twist-free solutions for the two cases are

$$ds^2 = r(\sin \sqrt{3}\theta \, dX^2 - 2 \cos \sqrt{3}\theta \, dX \, dY - \sin \sqrt{3}\theta \, dY^2) - dr^2 - r^2 \, d\theta^2, \qquad (4.1)$$

(Barnes 1976) and

$$ds^2 = 2 \, du \, dv - u^{-2n} v^{-2m} \, dx^2 - u^{-2m} v^{-2n} \, dy^2 \qquad (4.2)$$

where $m = \frac{1}{4}(\sqrt{3} - 1)$ and $n = -\frac{1}{4}(\sqrt{3} + 1)$ (Tariq and Tupper 1975).

If the coordinate transformations

$$\sqrt{2}u = \mathrm{i}r \, e^{\mathrm{i}\theta}, \qquad \sqrt{2}v = \mathrm{i}r \, e^{-\mathrm{i}\theta}$$

$$2^{1/4}x = X + \mathrm{i}Y, \qquad 2^{1/4}y = Y + \mathrm{i}X$$

are performed, then the metric (4.1) is transformed to (4.2). Thus the two metrics are distinct real slices of the same solution of the complexified Einstein–Maxwell field equations. By the complexified field equations I mean the usual set of spin coefficient Ricci identities, Maxwell equations and Bianchi identities for a real space–time with barred quantities replaced by tilded quantities plus the equations obtained by interchanging tilded and untilded quantities. The vectors l^i, n^i, m^i and $\tilde{m}^i$ are independent complex null vectors and tilded quantities are freed from being the complex conjugates of untilded ones. More details of the complexified field equations can be found in Fette *et al* (1976) and Flaherty (1976). It is perhaps worth emphasizing that the complex solution is not an H-space as it is neither Ricci-flat nor left- (or right-) conformally flat. The relationship between the two real slices is similar in many respects to that between the Schwarzschild solution and the B-metric (Flaherty 1976).

The existence of a complex analytic coordinate transformation relating the two real solutions is perhaps not surprising in view of the similarity between their derivations and the fact that they are related by the Sachs pseudo-tetrad transformation. A somewhat puzzling feature is that there is no solution for which m^i is divergence-free corresponding to the divergence-free solution of McLenaghan and Tariq (1975).

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